Phase Vector

In engineering, a sinusoidal signal
\[ x(t) = A \cos(2\pi f_0 t + \varphi) \]
is commonly described through a phasor
\[ \hat{x}(t) = A \exp(j(2\pi f_0 t + \varphi)) \]

It is useful to replace linear differential equations with algebraic equations.
Hilbert Transform (2/2)

The Hilbert inverse transform is simply obtained applying it twice

\[ \hat{x}(t) = \mathcal{H}\{\hat{x}(t)\} = x(t) \ast \frac{1}{\pi t} \ast \frac{1}{\pi t} \]

In Fourier domain

\[ \mathcal{F}\{\mathcal{H}\{\hat{x}(t)\}\}(f) = X(f) \frac{\text{sign}(f) \text{ sign}(f)}{j} = -X(f) \]

Back to time domain

\[ \mathcal{H}\{\hat{x}(t)\} = -x(t) \]

Equations for Hilbert (direct and inverse) transforms are

\[ \hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \]

\[ x(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau} d\tau \]

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Analytic Signal (1/3)

The analytic signal is a generalization of the concept of phasor to describe an arbitrary real-valued signal

\[ x(t) = \Re\{\hat{x}(t)\} \]

The following relation must hold as well

\[ \hat{x}(t) = \mathcal{H}\{x(t)\} \]

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Analytic Signal (2/3)

Consider \( x(t) \in \mathbb{R} \) and its Fourier transform

\[ X(f) = \mathcal{F}\{x(t)\}(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt \]

known to be Hermitian, i.e.

\[ X(f) = X^*(-f) \]

or else

\[ \begin{cases} |X(-f)| = |X(f)| \\ \angle X(-f) = -\angle X(f) \end{cases} \quad \begin{cases} \Re\{X(-f)\} = \Re\{X(f)\} \\ \Im\{X(-f)\} = -\Im\{X(f)\} \end{cases} \]

Knowing the behavior only for positive frequencies is enough

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Analytic Signal (3/3)

\( x(t) \) and \( \hat{x}(t) \), viewed as I/O of the following LTI system, represent the same information

\[ x(t) \xrightarrow{\mathcal{H}(f) = \mathcal{F}\{2u(f)\}} \hat{x}(t) \]

Also, from \( \hat{X}(f) = 2u(f)X(f) \) we get

\[ \hat{x}(t) = \mathcal{F}^{-1}\{2u(f)\} \ast x(t) = \mathcal{F}^{-1}\{1 + \text{sign}(f)\} \ast x(t) \]

\[ = \left( \delta(t) + \frac{1}{\pi t} \right) \ast x(t) = x(t) + \frac{1}{\pi t} \ast x(t) \]

\[ = x(t) + j\hat{x}(t) \]

\[ = x(t) + j\hat{x}(t) \]

\[ = x(t) + j\hat{x}(t) \]

\[ = \rho(t) \exp(j\phi(t)) \]

\( \hat{x}(t) \) is the analytic signal representing \( x(t) \)

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Properties

\[ x(t) = \sum_n a_n x_n(t) \quad \Rightarrow \quad \begin{cases} \hat{x}(t) = \sum_n a_n \hat{x}_n(t) \\ \hat{\hat{x}}(t) = \sum_n a_n \hat{\hat{x}}_n(t) \end{cases} \]

\[ z(t) = x(t) y(t) \quad \Rightarrow \quad \hat{z}(t) = x(t) \hat{y}(t) \]

\[ z(t) = x(t) \ast y(t) \quad \Rightarrow \quad \hat{\hat{z}}(t) = \frac{1}{2} \hat{x}(t) \ast \hat{\hat{y}}(t) = \hat{x}(t) \ast y(t) = x(t) \ast \hat{y}(t) \]

Correlation Properties

Using the standard I/O correlation relationships for LTI systems

\[ y(t) = h(t) \ast x(t) \]

\[ R_{yy}(\tau) = h(\tau) \ast h^*(-\tau) \ast R_{xx}(\tau) \]

we get

\[ R_{\hat{x}\hat{y}}(\tau) = R_{xy}(\tau) \]

\[ R_{\hat{x}\hat{y}}(\tau) = 2 R_{xy}(\tau) \]

\[ R_{\hat{\hat{x}}\hat{y}}(\tau) = 2 R_{xy}(\tau) + j 2 \hat{R}_{xy}(\tau) \]

\[ R_{xy}(\tau) = \frac{1}{2} \Re\{R_{\hat{\hat{x}}\hat{y}}(\tau)\} \]

PSD Properties (1/2)

\[ P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |x(t)|^2 dt \]

\[ = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |X_T(f)|^2 df \]

\[ = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} \left\{ \left| \frac{\text{sign}(f)}{j} \right|^2 |X_T(f)|^2 df \right\} \]

\[ = \lim_{T \to \infty} \frac{1}{T} \left\{ \int_{-T/2}^{+T/2} \left| X_T(f) \right|^2 df \right\} \]

\[ = P_x \]

PSD Properties (2/2)

\[ P_{\hat{x}} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |\hat{x}(t)|^2 dt \]

\[ = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} \left\{ 2 \left| u(f) \right|^2 |X_T(f)|^2 df \right\} \]

\[ = \int_{-T/2}^{+T/2} \left\{ 4 \left| u(f) \right|^2 \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2 df \right\} \]

\[ = 2 P_x \]

\[ = P_x + P_{\hat{x}} \]
Band-Pass Signals and Complex Envelope

A band-pass signal \( x(t) \) has significant spectral components within a finite range of frequencies not including the origin, i.e. \( 0 < B_1 < B_2 \)

\[
X(f) \approx 0 \quad \forall |f| \in \mathbb{R} - (B_1, B_2)
\]

The complex envelope (or equivalent low-pass signal) of \( x(t) \) is

\[
\tilde{x}(t) = \dot{x}(t) \exp(-j2\pi f_0 t)
\]

where \( f_0 \) is usually the centroid of the spectrum

\[
f_0 = \frac{\int_{-\infty}^{\infty} f X(f) df}{\int_{-\infty}^{\infty} X(f) df}
\]

I-Q Components and Rice Representation

In-Phase and Quadrature components of a band-pass signal \( x(t) \) are defined as

\[
x_I(t) = \Re\{\tilde{x}(t)\}
\]
\[
x_Q(t) = -\Im\{\tilde{x}(t)\}
\]

It is straightforward to show that

\[
\dot{x}(t) = \tilde{x}(t) \exp(+j2\pi f_0 t)
\]
\[
\ddot{x}(t) = x_I(t) - jx_Q(t)
\]

and finally

\[
x(t) = x_I(t) \cos(2\pi f_0 t) + x_Q(t) \sin(2\pi f_0 t)
\]

Some Relations

\[
\begin{align*}
\dot{x}(t) &= \rho(t) \exp(\theta(t)) \\
x(t) &= \rho(t) \cos(2\pi f_0 t + \theta(t)) \\
\dot{x}(t) &= x_I(t) \sin(2\pi f_0 t) - x_Q(t) \cos(2\pi f_0 t) \\
x_I(t) &= x(t) \cos(2\pi f_0 t) + \dot{x}(t) \sin(2\pi f_0 t) \\
x_Q(t) &= \dot{x}(t) \cos(2\pi f_0 t) - x(t) \sin(2\pi f_0 t)
\end{align*}
\]

\[
\begin{align*}
\rho(t) &= |\dot{x}(t)| = |\tilde{x}(t)| = \sqrt{x_I^2(t) + \dot{x}_I^2(t)} = \sqrt{x_Q^2(t) + \dot{x}_Q^2(t)} \\
\theta(t) &= \phi(t) - 2\pi f_0 t \\
\omega(t) &= \frac{d}{dt} \theta(t) \\
f(t) &= \frac{1}{2\pi} \omega(t)
\end{align*}
\]

Properties

\[
X(f) = \frac{X_I(f - f_0) + X_I(f + f_0)}{2} + \frac{X_Q(f - f_0) - X_Q(f + f_0)}{2j}
\]
\[
X_I(f) = \frac{X^-(f - f_0) + X^+(f + f_0)}{j}
\]
\[
X_Q(f) = \frac{X^-(f - f_0) - X^+(f + f_0)}{j}
\]

where \( s^-(t) = s(t) u(-t) \) and \( s^+(t) = s(t) u(t) \)
LT1 and Low-Pass Representation

Consider a pass-band LT1 system with impulse response $h(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

where the input $x(t)$ and the output $y(t)$ are band-pass signals

![Diagram of signal flow](image)

The relation between the complex envelopes is

$$\tilde{y}(t) = \tilde{y}(t)\exp(-j2\pi f_0 t) = \exp(-j2\pi f_0 t) \left( \tilde{x}(t) * \tilde{h}(t) \right)$$

$$= \exp(-j2\pi f_0 t) \left( \tilde{x}(t)\exp(+j2\pi f_0 t) * \tilde{h}(t)\exp(+j2\pi f_0 t) \right)$$

$$= \frac{1}{2} \int_{R} \tilde{h}(\tau)\tilde{x}(t-\tau)d\tau = \frac{1}{2} \tilde{x}(t) * \tilde{h}(t)$$

WSS Band-Pass Random Signals (2/2)

Consider a WSS band-pass random signal $x(t)$, then

- the complex envelope $\tilde{x}(t)$ is WSS
- the I-Q components $x_I(t)$ and $x_Q(t)$ are jointly WSS and orthogonal
- the I-Q components $x_I(t)$ and $x_Q(t)$ are incoherent if the PSD of $x(t)$ is symmetric around $f_0$

and finally

$$R_x(\tau) = R_{x_I}(\tau)\cos(2\pi f_0 \tau) + R_{x_Q}(\tau)\sin(2\pi f_0 \tau)$$

WSS Band-Pass Random Signals (1/2)

Consider a WSS band-pass random signal $x(t)$, then

$$R_x(\tau) = \mathbb{E}\{\tilde{x}(t)\tilde{x}^*(t-\tau)\}$$

$$= \mathbb{E}\{\tilde{x}(t)\exp(-j2\pi f_0 t)\tilde{x}^*(t-\tau)\exp(+j2\pi f_0 (t-\tau))\}$$

$$= R_x(\tau)\exp(-j2\pi f_0 \tau)$$

$$= 2 \left( R_x(\tau) + j\hat{R}_x(\tau) \right)\exp(-j2\pi f_0 \tau)$$

$$= 2 \left( R_x(\tau)\cos(2\pi f_0 \tau) + \hat{R}_x(\tau)\sin(2\pi f_0 \tau) \right)$$

$$+ j2 \left( \hat{R}_x(\tau)\cos(2\pi f_0 \tau) - R_x(\tau)\sin(2\pi f_0 \tau) \right)$$

i.e. the same relations refer now to the autocorrelation

$$R_{x_I}(\tau) = R_{x_Q}(\tau) = R_x(\tau)\cos(2\pi f_0 \tau) + \hat{R}_x(\tau)\sin(2\pi f_0 \tau)$$

$$R_{x_Ix_Q}(\tau) = -R_{x_I}(\tau)\sin(2\pi f_0 \tau) + \hat{R}_x(\tau)\cos(2\pi f_0 \tau)$$

An Interesting Example

Consider the following band-pass random signal $x(t)$, then

$$x(t) = A(t)\cos(2\pi f_0 t) + B(t)\sin(2\pi f_0 t)$$

where $A(t)$ and $B(t)$ are low-pass jointly WSS signals, then

- the signal is ciclostationary with average autocorrelation

$$R_x(\tau) = \frac{R_A(\tau) + R_B(\tau)}{2}\cos(2\pi f_0 \tau) - \frac{R_{AB}(\tau) - R_{BA}(\tau)}{2}\sin(2\pi f_0 \tau)$$

- the signal is WSS IFF $R_A(\tau) = R_B(\tau)$ and $R_{AB}(\tau) = -R_{BA}(\tau)$