Digital Communications
— Lecture 03 —
On-Off Keying

Pierluigi SALVO ROSSI
Department of Industrial and Information Engineering
Second University of Naples
Via Roma 29, 81031 Aversa (CE), Italy
homepage: http://wpage.unina.it/salvoros
email: pierluigi.salvorossi@unina2.it

Outline

1. System Model
2. Sufficient Statistic
3. Optimum Receiver
4. Performance

System Model

Binary memoryless modulation: \( M = 2, \ A = \{0, 1\}, \ \Pi = \{p, 1-p\} \)

Consider real-valued signals and noise

\[
\begin{align*}
s_0(t) &= 0 & s_1(t) &= p(t) & t \in [0, T) \\
\end{align*}
\]

AWGN channel: \( \mu_w(t) = 0, \ P_w(f) = \eta_0/2 \)

Signal Constellation

The signal space has dimension \( N = 1 \)

\[
\begin{align*}
\psi_1(t) &= \frac{1}{\sqrt{E_p}} p(t) \\
E_p &= \int_0^T p^2(t) dt
\end{align*}
\]

The signal constellation is

\[
\begin{align*}
s_0 &= <s_0, \psi_1> = 0 \\
s_1 &= <s_1, \psi_1> = \sqrt{E_p}
\end{align*}
\]
Received Signal

Two possible hypotheses in $t \in [0, T]$

- $H_0: r(t) = w(t)$
- $H_1: r(t) = p(t) + w(t)$

To represent the noise, thus the received signal, we need an infinite orthonormal set $\{\psi_n(t)\}_{n=1}^\infty$. Assume that $\psi_1(t) = p(t)/\sqrt{\varepsilon_p}$

$$r|H_0 = \begin{pmatrix} <w, \psi_1> \\ <w, \psi_2> \\ \vdots \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \end{pmatrix}$$

$$r|H_1 = \begin{pmatrix} <s + w, \psi_1> \\ <s + w, \psi_2> \\ \vdots \end{pmatrix} = \begin{pmatrix} \sqrt{\varepsilon_p} \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \end{pmatrix}$$

Sufficient Statistic (1/2)

Two considerations are crucial:

- Only the first component of the received vector is dependent on the two hypotheses
- The components of the received vector are statistically independent

The first component $r_1$, or simply $r$, represents a sufficient statistic for the considered detection problem, thus is the only to be computed

$$r|H_0 = w_1 \sim N\left(0, \frac{\eta_0}{2}\right)$$

$$r|H_1 = \sqrt{\varepsilon_p} + w_1 \sim N\left(\sqrt{\varepsilon_p} \cdot \frac{\eta_0}{2}\right)$$

Sufficient Statistic (2/2)

When transmitting the symbol 0 we observe a random variable with pdf

$$f_{r|H_0}(r) = \frac{1}{\sqrt{\pi \eta_0}} \exp \left(- \frac{r^2}{\eta_0}\right)$$

When transmitting the symbol 1 we observe a random variable with pdf

$$f_{r|H_1}(r) = \frac{1}{\sqrt{\pi \eta_0}} \exp \left(- \frac{(r - \sqrt{\varepsilon_p})^2}{\eta_0}\right)$$

Optimum Decision (1/2)

Decision is done via $\{\Omega_0, \Omega_1\}$ representing a partition of $\mathbb{R}$, i.e. $\Omega_0 \cup \Omega_1 = \mathbb{R}$ and $\Omega_0 \cap \Omega_1 = \emptyset$, with

$$r \in \Omega_0 \rightarrow rx \ 0 \quad \text{,} \quad r \in \Omega_1 \rightarrow rx \ 1$$

Each partition leads to a different error probability

$$P_e \triangleq \Pr \left\{ \{tx \ 0, \ rx \ 1\} \cup \{tx \ 1, \ rx \ 0\} \right\}$$

$$= \Pr \left\{ \{rx \ 1|tx \ 0\} \right\} + \Pr \left\{ \{rx \ 0|tx \ 1\} \right\} (1 - p)$$

$$= \Pr \left\{ r|H_0 \in \Omega_1 \right\} + \Pr \left\{ r|H_1 \in \Omega_0 \right\} (1 - p)$$

$$= \int_{\Omega_0} pf_{r|H_0}(r) dr_1 + \int_{\Omega_1} (1 - p) f_{r|H_1}(r) dr_1$$

$$= \int_{\Omega_0} pf_{r|H_0}(r) dr_1 + \int_{\Omega_1} (1 - p) f_{r|H_1}(r) dr_1$$

$$= p - \int_{\Omega_0} pf_{r|H_0}(r) - (1 - p) f_{r|H_1}(r) dr_1$$
Optimum Decision (2/2)

The optimum decision leads to the minimum Pr(e), thus

\[ \Omega_0 = \{ r \in \mathbb{R} : p_{r|H_0}(r) > (1-p)f_{r|H_1}(r) \} \]
\[ \Omega_1 = \{ r \in \mathbb{R} : p_{r|H_0}(r) < (1-p)f_{r|H_1}(r) \} \]

with

\[ \lambda : \quad p_{r|H_0}(\lambda) = (1-p)f_{r|H_1}(\lambda) \]

Receiver Architecture (1/2)

First option

![First option diagram]

Second option

![Second option diagram]

Optimum Threshold

\[ \lambda : \quad p_{r|H_0}(\lambda) = (1-p)f_{r|H_1}(\lambda) \]
\[ \frac{p}{\sqrt{\eta_0}} \exp \left( -\frac{\lambda^2}{\eta_0} \right) = \frac{1-p}{\sqrt{\eta_0}} \exp \left( -\frac{(\lambda - \sqrt{E_p})^2}{\eta_0} \right) \]

The optimum threshold depends on the energy, on the noise PSD and on the a-priori probabilities:

\[ \lambda = \frac{\sqrt{E_p}}{2} + \frac{\eta_0}{2\sqrt{E_p}} \log \left( \frac{p}{1-p} \right) \]

In case of transmission with equiprobable symbols, i.e. \( p = 1/2 \), the optimum decision is a minimum distance decision:

\[ \lambda = \frac{\sqrt{E_p}}{2} \]
\[ P_e = 1 - P_c = 1 - p \Pr(c|H_0) - (1 - p) \Pr(c|H_1) \]
\[ = 1 - p \Pr(r|H_0 < \lambda) - (1 - p) \Pr(r|H_1 > \lambda) \]
\[ = 1 - p \int_{-\infty}^{\lambda} f_{r|H_0}(r)dr - (1 - p) \int_{\lambda}^{+\infty} f_{r|H_1}(r)dr \]
\[ = 1 - p \Pr(N(0, \frac{\eta_0}{2}) < \lambda) - (1 - p) \Pr(N(\sqrt{\varepsilon_p}, \frac{\eta_0}{2}) > \lambda) \]
\[ = 1 - p \left( 1 - Q\left( \frac{\lambda}{\sqrt{\eta_0/2}} \right) \right) - (1 - p)Q\left( \frac{\lambda - \sqrt{\varepsilon_p}}{\sqrt{\eta_0/2}} \right) \]

In case of equiprobable symbols

\[ P_e = Q\left( \sqrt{\frac{\varepsilon_p}{2\eta_0}} \right) \]